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The onset of convective instability in a horizontal fluid layer subjected to a constant heat flux from below

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Abstract

The onset of buoyancy-driven convection in an initially isothermal, quiescent fluid layer confined between the two infinite horizontal plates is analyzed by using propagation theory. For the system heated rapidly from below with a constant heat flux, the dimensionless critical time τ_c to mark the onset of convective instability is presented as a function of the Rayleigh number and the Prandtl number. The present predictions show that τ_c decreases with increasing Prandtl number for a given Rayleigh number. Available experimental data for water and silicone oil indicate that visible motion is detected first and then, an obvious deviation of the heated surface temperature from its conduction solution occurs. The difficulty of defining the detection time of visible motion is discussed in comparison with experiments.
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Keywords: Buoyancy-driven instability; Constant flux heating; Critical time; Propagation theory; Rayleigh number

1. Introduction

Natural convection is encountered in nature and in a number of engineering problems involving heat transfer. A well-known example is Bénard convection. When an initially quiescent fluid layer confined between two horizontal plates is heated rapidly from below, the basic temperature profile of heat conduction develops with time and buoyancy-driven convection sets in at a certain time. In this transient system, an important issue is the prediction of the critical time t_c that marks the onset of convective instability. This transient problem, may be called an extension of classical Rayleigh–Bénard problems. Morton [1], Yang and Choi [2], Tan and Thorpe [3], Foster [4], and Jhaveri and Homsy [5] conducted the related instability analyses by using the frozen-time model, propagation theory, maximum-Rayleigh-number criterion, amplification theory, and stochastic model, respectively. The first two models are based on linear theory and yield the critical time to mark the onset of a fastest growing mode of convective instability. The third model is the simplest one, which is based on the conduction temperature, and the last two models require the initial conditions at the heating time $t = 0$. These three models deal with the detection time of manifest convection. However, there is confusion about the characteristic times, which are the onset time of growing instabilities (t_c), the time when the first visible motion can be detected (t_D), and the detection time of manifest convection (t_m).

Experimental observations by Nielsen and Sabersky [6], Chu [7], and Goldstein and Volino [8] show that,

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Nomenclature

a	dimensionless horizontal wavenumber, $\sqrt{a_x^2 + a_y^2}$
a^*	modified wavenumber, $a\tau^{1/2}$
d	depth of fluid layer (m)
g	gravitational acceleration constant (ms^{-2})
k	thermal conductivity (Wm^{-1} K^{-1})
Nu	Nusselt number, averaged value of $q_w d / k(T_w - T_i)$
Pr	Prandtl number, ν/α
q_w	heat flux at the bottom wall (Wm^{-2})
Ra_q	Rayleigh number based on the heat flux, $g\beta q_w d^4/k\alpha\nu$
Ra^*	modified Rayleigh number, $Ra_q \tau^2$
r_0	temporal growth rate of the conduction field
r_1	temporal growth rate of the perturbed temperature field
T	temperature (K)
t	time (s)
W_1	perturbed vertical velocity (ms^{-1})
w_1	dimensionless vertical velocity disturbance, $W_1 d/\alpha$
Z	vertical distance (m)
x, y, z	dimensionless Cartesian coordinates, $(X, Y, Z)/d$

Greek symbols

α	thermal diffusivity (m^2 s^{-1})
β	volumetric thermal expansion coefficient (K^{-1})
Δ_T	thermal penetration depth (m)
δ_T	dimensionless thermal penetration depth, Δ_T/d
∇_1^2	horizontal Laplacian operator, $\partial/\partial x^2 + \partial/\partial y^2$
θ_0	dimensionless basic temperature, $k(T - T_i)/q_w d$
θ_1	dimensionless temperature disturbance, $T_1 g\beta d^3/\alpha\nu$
ν	kinematic viscosity (m^2 s^{-1})
τ	Dimensionless time, $\alpha t/d^2$
ζ	similarity variable, $z/\sqrt{\tau}$

Subscripts

c	critical state
D	detection
i	initial state
m	manifest convection
rms	root-mean-square quantity
0	basic state
1	perturbed state

Superscript

*	amplitude function

with rapid heating, cell-like patterns like Bénard cells appear suddenly over the heated bottom surface at a certain time t_D, subject to an appropriate measurement technique. Chu's experimental results show that the first visible motion is detected at the characteristic time t_D earlier than t_m. For the present problem of a constant heat flux, the characteristic time t_m is the time to ensure an obvious deviation of the heated wall temperature T_w from its conduction solution. Here the critical time t_c to mark the onset of convective instability is defined as that time when a fastest growing, single mode of infinitesimal disturbances sets in. During the period $0 \leqslant t < t_c$ the temporal growth rate of temperature disturbances will be less than that of the conduction temperature. For $t \geqslant t_c$ incipient disturbances will grow faster and at $t = t_m$ a change in T_w from the conduction solution will be detected experimentally. This phenomenon is reasonably represented by the analysis based on propagation theory among the above models even though it is a rather simple one.

Propagation theory deals with instability problems of developing, nonlinear temperature profiles for large Rayleigh numbers. In the transient conduction system it is assumed that at $t = t_c$ infinitesimal temperature disturbances are propagated mainly within the thermal penetration depth Δ_T. Therefore, with the length scaling factor Δ_T all the variables and parameters having the length scale are rescaled. In a usual deep-pool system of $\Delta_T \propto \sqrt{\alpha t}$, the most important parameter becomes the time-dependent Rayleigh number, which results by replacing the length scale in the Rayleigh number with Δ_T. Here α is the thermal diffusivity. Under normal mode analysis the self-similar transformations are forced, and the critical time to mark the onset of a fastest growing mode of convective instability is obtained for a given Rayleigh number and Prandtl number. The resulting stability criteria agree with experimental data reasonably well in other transient systems [2,9–12].

Here we analyze the instability problem of an initially isothermal, quiescent fluid layer with the bottom boundary heated uniformly with a constant heat flux q_w starting from $t = 0$. For the present system the instability criteria will be obtained based on propagation theory and compared with available experimental data and other theoretical results.

2. Stability analysis

2.1. Disturbance equations

The system considered here is a Newtonian fluid layer having a constant initial temperature T_i, which is placed between the two infinite horizontal plates. For time $t \geqslant 0$, the fluid layer of depth d is heated from below with a constant heat flux q_w, and its upper boundary is kept at T_i. The schematic diagram of the basic system of pure conduction is shown in Fig. 1. For a high q_w, buoyancy-driven convection will set in at a certain time. The Boussinesq approximation is employed to describe the buoyancy force.

Under linear stability theory the infinitesimal disturbances caused by incipient convective motion are, in the usual dimensionless form, expressed [2,4]

$$\left\{\frac{1}{Pr}\frac{\partial}{\partial\tau} - \nabla^2\right\}\nabla^2 w_1 = \nabla_1^2\theta_1, \tag{1}$$

$$\frac{\partial\theta_1}{\partial\tau} + Ra_q w_1 \frac{\partial\theta_0}{\partial z} = \nabla^2\theta_1, \tag{2}$$

where τ, w_1, θ_1, θ_0 and ∇_1^2 denote the dimensionless forms of the time, the vertical velocity disturbance, the temperature disturbance, the basic temperature, and the horizontal Laplacian, respectively. The dimensionless Cartesian coordinates (x, y, z) have the scale of d and the subscripts '1' and '0' denote the perturbed state and the basic state, respectively. The proper boundary conditions are given by

$$w_1 = \frac{\partial w_1}{\partial z} = \frac{\partial\theta_1}{\partial z} = 0 \quad \text{at } z = 0, \tag{3a}$$

$$w_1 = \frac{\partial w_1}{\partial z} = \theta_1 = 0 \quad \text{at } z = 1, \tag{3b}$$

wherein the condition of no slip is applied to the two rigid boundaries. As stated above, the lower boundary condition is $q_w = \text{constant}$ and the upper one is $T_i = \text{constant}$. The important parameters to describe the present system, the Prandtl number Pr and the Rayleigh number Ra_q (sometimes called the dimensionless heat flux), are defined as

$$Pr = \frac{\nu}{\alpha}, \quad Ra_q = \frac{g\beta q_w d^4}{k\alpha\nu}. \tag{4}$$

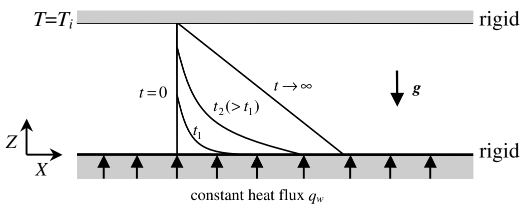

Fig. 1. Sketch of the basic conduction state considered here.

Here ν, g, β and k represent the kinematic viscosity, the gravitational acceleration constant, the thermal expansion coefficient, and the thermal conductivity, respectively. In the case of very slow heating the basic temperature profile eventually becomes linear and time-independent, and the critical Rayleigh number reaches the well-known value of $Ra_{q,c} = 1296$. This critical value is independent of Pr. But in a rapidly heated system with large Ra_q, the resulting transient stability problem is complicated. For a given Pr and Ra_q, the critical time τ_c to mark the onset of convective instability should be found by using Eqs. (1)–(3).

2.2. Basic state of heat conduction

Before convective motion sets in, the heat transfer is dominated by heat conduction and the dimensionless temperature profile is represented by

$$\frac{\partial\theta_0}{\partial\tau} = \frac{\partial^2\theta_0}{\partial z^2}, \tag{5}$$

with the following initial and boundary conditions,

$$\theta_0 = 0 \quad \text{at } \tau = 0 \quad \text{and} \quad z = 1, \tag{6a}$$

$$\frac{\partial\theta_0}{\partial z} = -1 \quad \text{at } z = 0. \tag{6b}$$

The exact solution for the dimensionless temperature θ_0 is obtained by using the Fourier and the Laplace transforms of Eqs. (5) and (6) to yield

$$\bar{\theta}_0 = 1 - z - 2\sum_{m=1}^{\infty}\frac{1}{\mu_m}\cos(\mu_m z)\exp\left(-\mu_m^2\tau\right), \tag{7a}$$

$$\theta_0 = \sqrt{4\tau}\sum_{m=0}^{\infty}(-1)^m\left\{\operatorname{ierfc}\left(\frac{m}{\sqrt{\tau}} + \frac{\zeta}{2}\right) - \operatorname{ierfc}\left(\frac{m+1}{\sqrt{\tau}} - \frac{\zeta}{2}\right)\right\}, \tag{7b}$$

where $\bar{\theta}_0(\tau, z) = \theta_0(\tau, \zeta)$, $\mu_m = (m - 1/2)\pi$, $\zeta = z/\tau^{1/2}$, and $\operatorname{ierfc}(\cdot)$ is the integral of the complementary error function.

For deep-pool systems with $\Delta_T \propto \sqrt{\alpha t}$, the Leveque-type solution is given by

$$\theta_0 = \sqrt{4\tau}\left\{\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\zeta^2}{4}\right) - \frac{\zeta}{2}\operatorname{erfc}\left(\frac{\zeta}{2}\right)\right\} = \sqrt{4\tau}\operatorname{ierfc}\left(\frac{\zeta}{2}\right), \tag{8}$$

which is in good agreement with Eq. (7) for $\tau < 0.05$, as shown in Fig. 2. With this solution the following relationship is obtained in coordinates τ and ζ with $\partial\theta_0^*/\partial\tau \cong 0$:

$$\frac{\partial\bar{\theta}_0^*}{\partial\tau} = -\frac{\zeta}{2\tau}D\theta_0^* \quad \text{for } \tau < 0.05, \tag{9}$$

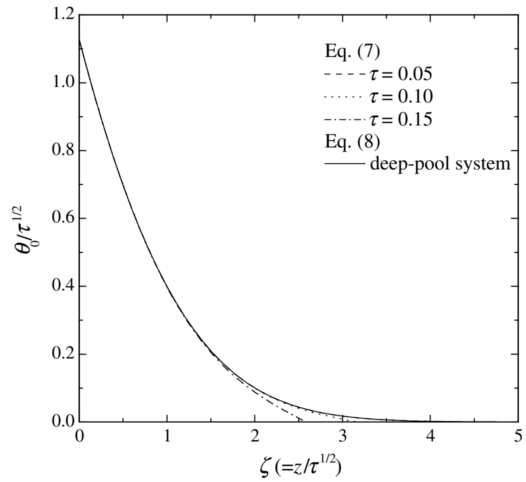

Fig. 2. Comparison of Eq. (8) with Eq. (7). For $\tau \leqslant 0.05$ the difference between two equations becomes negligible.

where $(\bar{\theta}_0^*, \theta_0^*) = (\bar{\theta}_0(\tau, z), \theta_0(\tau, \zeta))/\sqrt{\tau}$, and $D = d/d\zeta$. Since we are concerned with the deep-pool case of large Ra_q and small τ, the above Leveque-type solution (8) is primarily used for the stability analysis.

2.3. Stability equations

At the critical time to mark the onset of convective instability, disturbances are assumed to exhibit horizontal periodicity, and normal mode analysis is employed. Then the perturbed quantities can be expressed as

$$[w_1(\tau, x, y, z), \theta_1(\tau, x, y, z)] = [w_1^*(\tau, z), \theta_1^*(\tau, z)] \times \exp[\mathrm{i}(a_x x + a_y y)], \tag{10}$$

where $\mathrm{i} = \sqrt{-1}$, and the horizontal wavenumber "a" is given by $a = \left[a_x^2 + a_y^2\right]^{1/2}$. Propagation theory, employed to find the onset time of convective instability, i.e., the critical time t_c, is based on the assumption that in deep-pool systems the infinitesimal temperature disturbances are propagated mainly within the thermal penetration depth Δ_T at the time of the onset of convective instability. Based on this assumption, the following amplitude relation is obtained in dimensionless form from Eqs. (1) and (10):

$$\frac{w_1^*}{\theta_1^*} \sim \delta_T^2 \sim \tau, \tag{11}$$

from the balance between viscous and buoyancy terms. Here $\delta_T (= \Delta_T/d = 3.21\tau^{1/2})$ is the usual dimensionless thermal penetration depth defined by writing $\theta_0^*|_{z=\delta_T}/\theta_0^*|_{z=0} = 0.01$. Eqs. (2) and (10) yield

$$Ra^*(w_1^*/\tau)D\theta_0^* \sim \theta_1^*, \tag{12}$$

where $Ra^* = Ra_q\tau^2$. There are many possible forms of dimensionless amplitude functions of disturbances that satisfy relations (11) and (12), for example,

$$\left[w_1^*(\tau, z), \theta_1^*(\tau, z)\right] = \left[\tau^{n+1}\bar{w}_1^*(\tau, z), \tau^n\bar{\theta}_1^*(\tau, z)\right]. \tag{13}$$

If the related process is still conduction-dominant with $Ra^* =$ constant at small time, it is probable that $\bar{\theta}_1^*(\tau, z) = \theta^*(\zeta)$, $\bar{w}_1^*(\tau, z) = w^*(\zeta)$ and $\partial\theta_1^*/\partial\tau = -(\zeta/(2\tau))D\theta^*$. This parallels Eq. (9) and means that the amplitude function of temperature disturbances has a form similar to Eq. (8) for small τ. The result of $Ra^* =$ constant is also obtained using the stochastic model [5,13] and the amplification theory [4,14].

The above scaling analysis for a deep-pool system of $\delta_T \propto \tau^{1/2}$ suggests relations of the form $w_1^* = \tau^{3/2}w^*(\zeta)$ and $\theta_1^* = \tau^{1/2}\theta^*(\zeta)$ with $n = 1/2$. This condition shows that the temporal growth rate of the perturbed temperature field (r_1) is equal to that of the conduction field (r_0). Here r_0 and r_1 are defined, based on Eqs. (7) and (10), respectively

$$r_0 = \frac{1}{\theta_{0,\mathrm{rms}}}\frac{\mathrm{d}\theta_{0,\mathrm{rms}}}{\mathrm{d}\tau}, \tag{14a}$$

$$r_1 = \frac{1}{\theta_{1,\mathrm{rms}}}\frac{\mathrm{d}\theta_{1,\mathrm{rms}}}{\mathrm{d}\tau}, \tag{14b}$$

where the subscript 'rms' denotes the root-mean-square quantity, i.e., $(\cdot)_{\mathrm{rms}} = \sqrt{\left(\int_V (\cdot)^2 \mathrm{d}V\right)/V}$, and V is the volume of the system considered. Eq. (8) yields $r_0 = 3/(4\tau)$. For $\tau < \tau_c$ temperature disturbances are conceptually so small in comparison with the average quantities of conduction temperature that with $r_1 < r_0$ the latter ones are still much larger. Therefore the system is considered stable when r_1 is less than r_0 and the neutral stability curve is obtained with $r_0 = r_1 = 0$ for $\tau_c \to \infty$. The related concept is well illustrated in the numerical work of Choi et al. [15]. In all the previous studies based on propagation theory [2,9–12] the stability analyses were conducted with $n = 0$, which is the condition required to satisfy the above constraint on the temporal growth rates in isothermally heated systems. It is noted that the stability criteria with $n = 0$ are of the same order as those with $n = 1/2$, and a lower n-value yields a smaller τ_c-value. The dimensionless time τ plays dual roles of time and length.

Now, with $n = 1/2$ the self-similar stability equations are obtained from Eqs. (1) and (2) for small τ

$$\left\{\left(D^2 - a^{*^2}\right)^2 + \frac{1}{2Pr}\left(\zeta D^3 - D^2 - a^{*^2}\zeta D + 3a^{*^2}\right)\right\}w^* = a^{*^2}\theta^*, \tag{15}$$

$$\left\{D^2 + \frac{1}{2}\zeta D - \left(a^{*^2} + \frac{1}{2}\right)\right\}\theta^* = Ra^* w^* D\theta_0^*, \tag{16}$$

by treating Ra^* and a^* as eigenvalues. Here $a^* = a\tau^{1/2}$. The appropriate boundary conditions are transformed from Eq. (3) to give

$$w^* = Dw^* = D\theta^* = 0 \quad \text{at } \zeta = 0, \tag{17a}$$

$$w^* = Dw^* = \theta^* = 0 \quad \text{as } \zeta \to \infty. \tag{17b}$$

For a given Pr the minimum value of Ra^* should be found in a plot of Ra^* vs a^* using the principle of the exchange of stabilities. In other words, the minimum value of τ, i.e., τ_c and its corresponding wavenumber a_c are obtained for a given Pr and Ra_q. Since time is frozen by letting $\partial(\cdot)/\partial\tau \equiv 0$ under the frame of amplitude coordinates τ and ζ instead of τ and z (see Eqs. (7) and (8)), propagation theory may be called the relaxed frozen-time model by implicitly treating time as the parameter. However, it considers the time dependency.

In the conventional frozen-time model the terms involving $\partial(\cdot)/\partial\tau$ in Eqs. (1) and (2) are neglected in amplitude coordinates τ and z. This results in $\left(D^2 - a^{*2}\right)^2 w^* = a^{*2}\theta^*$ and $\left(D^2 - a^{*2}\right)\theta^* = Ra^* w^* D\theta_0^*$ instead of Eqs. (15) and (16). The resulting stability criteria become independent of Pr.

3. Solution procedure

The stability Eqs. (15)–(17) were solved by employing the outward shooting scheme. In order to integrate these stability equations the appropriate values of D^2w^*, D^3w^* and θ^* at $\zeta = 0$ are assigned for a given Pr and a^*. Since the stability equations and their boundary conditions are all homogeneous, the value of $D^2w^*(0)$ can be given arbitrarily and the value of the parameter Ra^* is assumed. After the initialization this eigenvalue problem can be solved numerically by following the method illustrated by Kim et al. [12]. In the limiting case of $Pr \to \infty$ the stability equations are reduced to simpler forms because the inertia terms involving Pr in Eq. (15) are negligible. The marginal stability curve is shown in Fig. 3, wherein $Ra_c^* = 31.03$ and $a_c^* = 0.61$. From these values τ_c and a_c are obtained for a given Ra_q.

For $\tau_c > 0.05$, Eqs. (15) and (16) are retained, and Eq. (7b) is used. In boundary condition (7b) the upper boundary $\zeta \to \infty$ is replaced by $z = 1$, i.e., $\zeta = 1/\tau_c^{1/2}$ and in Eqs. (15) and (16) Ra^* and a^* are replaced by $Ra_q\tau_c^2$ and $a\tau_c^{1/2}$. Also, in Eq. (7b), τ is replaced with τ_c but ζ is maintained. Since τ_c is the fixed parameter, the resulting stability equations are a function of ζ only and the physics of Eqs. (9) and (11) is still satisfied. For a given Pr and τ_c the minimum Ra_q-value and its corresponding wavenumber a_c are obtained. For the frozen-time model the solution procedure is almost the same as above.

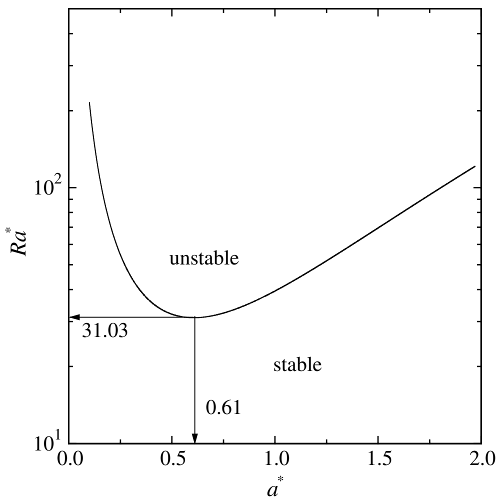

Fig. 3. Marginal stability curve for deep-pool systems of $Pr \to \infty$.

4. Results and discussion

For the deep-pool system heated rapidly from below with q_w = constant, the critical conditions predicted by propagation theory are summarized in Table 1 and Fig. 4. It is believed that for a given Ra_q and Pr a fastest growing, single mode of infinitesimal disturbances would set in at $\tau = \tau_c$ with $a = a_c$. The present critical time τ_c in Table 1 yields the correlation, based on the case of $Pr \to \infty$.

$$\tau_c = 5.57\left[1 + \left(\frac{0.55}{Pr}\right)^{5/8}\right]^{4/5} Ra_q^{-1/2}$$

$$\text{for } \tau_c < 0.05 \text{ with } Pr \geqslant 0.01, \tag{18}$$

which has an error bound of 6%. This correlation shows that τ_c decreases with an increase in Ra_q and also Pr. The Pr-effect becomes pronounced for $Pr < 1$, which means the inertia terms make the system more stable. For $Pr > 7$, τ_c and a_c are almost independent of Pr, as shown

Table 1
Present critical conditions obtained from propagation theory for deep-pool systems of various Pr-values

Pr	0.01	0.1	0.7	1	7	10	100	∞
$\tau_c Ra_q^{1/2}$	46.4	17.0	8.85	8.12	6.11	5.96	5.61	5.57
$a_c\tau_c^{1/2}$	0.94	0.92	0.82	0.80	0.67	0.66	0.61	0.61

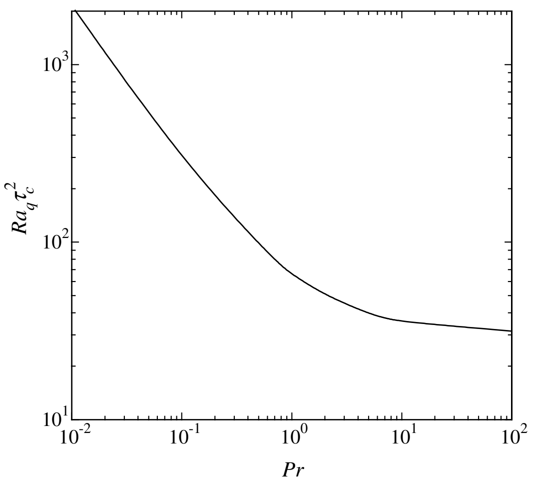

Fig. 4. Effect of Pr on $Ra_q\tau_c^2$ for deep-pool systems.

in Fig. 4, and therefore, the following critical conditions of $Pr \to \infty$ may be used up to moderate Pr-values

$$\tau_c = 5.57 Ra_q^{-1/2}, \tag{19a}$$

$$a_c = 0.61\tau_c^{1/2}. \tag{19b}$$

For the whole time domain the stability criteria from propagation theory are shown in Fig. 5. The results obtained from the conventional frozen-time model are also indicated. With increasing τ_c they approach the well-known critical conditions of $Ra_{q,c} = 1296$ and $a_c = 2.55$ since the basic temperature profile becomes linear. The frozen-time model yields the lower bounds of τ_c and a_c, as shown in the figures. It is known that the terms involving $\partial(\cdot)/\partial\tau$ in Eqs. (1) and (2) stabilize the system.

For the present system of q_w = constant, Nielsen and Sabersky [6] observed the change in the temperature gradient of silicone oil ($Pr = 45, 896, 4770$) on a shadowgraph. In their experimental range of $9.2 \times 10^2 \leqslant Ra_q \leqslant 1.9 \times 10^7$ their data points are well-correlated by

$$\tau_m \cong 19 Ra_q^{-1/2} \quad \text{for } Ra_q \geqslant 10^4, \tag{20}$$

as shown in Fig. 6. Here τ_m is the characteristic time to ensure an obvious deviation of T_w from the conduction solution. It is interesting that with the present τ_c-values $\tau_m \cong 3.2\tau_c$ over the whole time domain.

Kim and Kim [13] reported results of a Monte-Carlo simulation by employing random fluctuations, starting

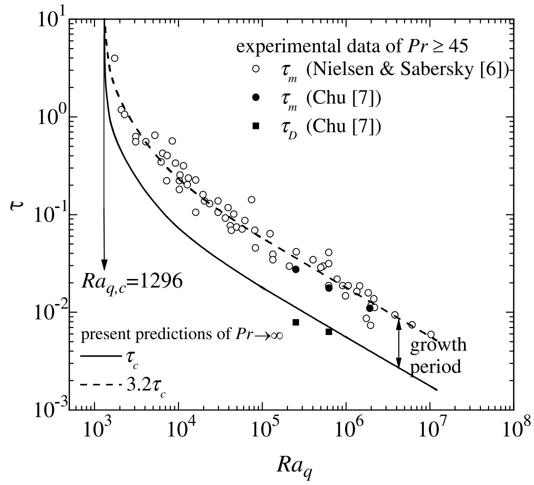

Fig. 6. Comparison of characteristic times with available experimental data.

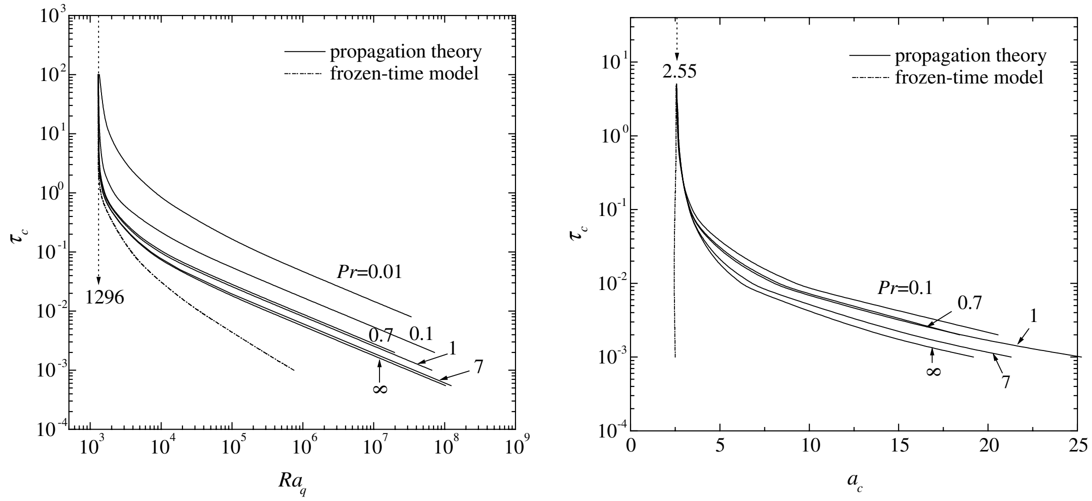

Fig. 5. Effect of Pr on stability criteria: (a) τ_c vs. Ra_q and (b) τ_c vs. a_c.

from $t = 0$ and fixing the wavenumber, similar to the work of Jhaveri and Homsy [5]. With the fastest growing, particular wavenumber they solved their nonlinear evolution equations and defined t_m as that time when the Nusselt number $Nu(= \langle q_w d/k(T_w - T_i)\rangle)$ attains the minimum in the plot of Nu vs. τ. Here $\langle\ \rangle$ denotes the horizontal averaged value. Based on this undershoot time of Nu, they obtained correlations: $\tau_m Ra_q^{1/2} = 20.2$ for $Pr = 45$, 25.6 for $Pr = 7$, and 49.5 for $Pr = 0.7$. Based on the present τ_c-values in Table 1, these values correspond to $\tau_m/\tau_c = 3.6$, 4.2 and 5.6, respectively. This means that for a given Ra_q, τ_m decreases with increasing Pr. Tan and Thorpe [3] suggested a simple model assuming that at the onset of manifest convection the following relation is satisfied, maximum of $(g\beta Z^4/\nu k)\partial T_0/\partial Z = 1296$. This is based on the relation of $Ra_{q,c} = 1296$. This results in $\tau_m = 20.7Ra_q^{-1/2}$ and is independent of Pr. This model becomes invalid for small Pr. It is interesting to note that a common relation is seen in all the above theoretical results, that is, for small times $\delta_T \propto \tau^{1/2}$ and $Ra_q\delta_T^4 \sim$ constant at τ_c or τ_m, as shown in the scale relations (11) and (12). For large Pr, the results are comparable to Eq. (20).

The above phenomena are illustrated in detail by Goldstein and Volino [8]. Their experimental data for water are converted and listed in Table 2. They measured the detection time of convection (t_D) in the water layer ($Pr \cong 7$) using the three methods: interferometer visualization (data sets No. 1–6), liquid crystal visualization (data set No. 7), and thermocouple data (data set No. 8). Here t_D denotes the time at which the first visible motion is detected by each measurement technique. They averaged the data from each experimental set of 23–66 runs. The numbers associated with interferometer visualization, Nos. 1 through 6, correspond to the first, second, third, fourth, fifth, sixth largest amplitudes, respectively. The values of $Ra_D^*\left(= Ra_q\tau_D^2\right)$ and $a_D^*\left(= a\tau_D^{1/2}\right)$ in the table have been obtained based on the present τ_c-values ($Ra_q^{1/2}\tau_c = 6.11$ for $Pr = 7$ in Table 1). The data sets 1 through 6 show that $\tau_D = (1.80\text{–}2.79)\tau_c$. They reported that experimental τ_D-values are strongly dependent upon boundary imperfections. The converted value $a_D^*(\tau_c/\tau_D)^{1/2}$, i.e., $a_c\tau_c^{1/2}$ is almost 0.8, which is larger than 0.61 in Table 1. This means that the average size of incipient cells is smaller than the present prediction but it is almost constant during the growth period of $1.80\tau_c \leqslant \tau \leqslant 2.79\tau_c$. The a_D^*-values of No. 7 correspond to the wavenumbers perpendicular and parallel to the optical windows for a liquid crystal sheet below the heated surface, respectively, which agree well with the results reported in Table 1. Considering the standard deviation in a_D^* of 19–26% listed in Table 2, it may be stated that the present predictions given in Table 1 represent water data to a certain degree. Data sets 7 and 8 lead to $\tau_D \cong 3.4\tau_c$ and $\tau_D \cong 21Ra_q^{-1/2}$, respectively, which compare well with Eq. (20).

Chu [7] observed the first motion at $t \cong 0.3t_m$ for $Ra_q = (2.56\text{–}6.37) \times 10^5$ by photographing paths of seeded particles in syrup ($Pr = 4.5 \times 10^5$), and he reported that convection usually starts at the edge of an enclosure. His data are shown in Fig. 6 and his τ_D-values are located near the present τ_c-values. His data yield the relation of $\tau_m \cong 3.3\tau_D$, which is almost the same as the present one, i.e., $\tau_m \cong 3.2\tau_c$ with $\tau_D = \tau_c$.

The above experimental and theoretical results for large-Pr systems suggest that a fastest growing mode of instabilities, which sets in at $t = t_c$, will grow with time until manifest convection appears near the lower surface and is detected at $t = t_m$. In connection with the growth period (from the onset time of convective instabilities to the detection time of manifest convection), Foster [14] suggested the relation of $t_m \cong 4t_c$. This relation is supported by the present study approximately. It seems evident that during the period $t_c \leqslant t \leqslant t_m$, the cell size is almost constant, but its vertical growth continues. During this growth period, convective motion seems relatively weak since the related heat transport is well represented by the conduction state. Its detection time depends, to a certain degree, on the measurement method used, and linear theory may be applied. If boundary imperfections exist, multiple-mode instabilities will arise. Therefore, well-defined single-mode cells seem to appear rarely in experiments with rapid heating.

Table 2
Values of $a_c\tau_c^{1/2}$ converted from Goldstein and Volino's [8] water data at the dimensionless detection time τ_D

Number	τ_D/τ_c	Ra_D^*	a_D^*	$a_c\tau_c^{1/2}$
1	1.80	121	1.03	0.77
2	2.09	163	1.13	0.78
3	2.32	201	1.18	0.77
4	2.50	234	1.23	0.78
5	2.69	271	1.30	0.79
6	2.79	291	1.34	0.80
7	3.37	425	1.19, 1.25	0.65, 0.68
8	3.39	430	–	–

(Basis) $\tau_c Ra_q^{1/2} = 6.11$ for $Pr = 7$ from Table 1; $Ra_D^* = Ra_q\tau_D^2 = Ra_c^*(\tau_D/\tau_c)^2$; $a_D^* = a\tau_D^{1/2}$; $a_D^*(\tau_c/\tau_D)^{1/2} = a_c\tau_c^{1/2}$.

The present stability criteria are approximate ones because propagation theory forces the self-similar transformation. For more refined analysis numerical simulations are required. In addition, new measures to mark the onset of convective instability and manifest convection need definition. The onset time of intrinsic instability will be the earliest time at which the fastest growing mode of infinitesimal motion sets in. It is expected that the incipient instability will grow to manifest convection at the earliest time. For the present constant-heat-flux system, the time to mark an obvious deviation of the heated surface temperature from the conduction solution (t_m, e.g. the undershoot time) can be a definitive measure of the detection time of manifest convection even though earlier motion can be detected. For $t > t_m$, the averaged horizontal size of cells will increase significantly due to merging of cells with time until flow becomes fully developed, as shown in experimental results of Goldstein and Volino [8]. Linear theory loses its validity under these conditions.

5. Conclusion

The critical condition to mark the onset of convective instability in an initially quiescent, horizontal fluid layer heated from below with a constant heat flux has been analyzed by using propagation theory. This model has been improved by considering the temporal growth rates of the unperturbed and the perturbed temperature field. The τ_c-values have been obtained as a function of Ra_q and Pr. It is believed that the onset time of intrinsic instability, i.e., the earliest time when the above two growth rates become the same, is located near the present τ_c-values. A more refined analysis is now being conducted numerically.

The present study shows that the characteristic time to mark the onset of visible motion requires its careful definition. In the present constant-heat-flux system the characteristic time t_m should be considered to be the time at which an obvious deviation of the heated surface temperature from its conduction state occurs, for example, the undershoot time in the plot of Nu vs. τ. For $Pr \geqslant 7$, available experimental and theoretical results yield the approximation $t_m \cong 3.2t_c$. For $t \leqslant t_m$, the velocity magnitude is relatively small and therefore, the heat transfer rate is not enhanced significantly. It is interesting that the present predictions based on propagation theory cover the whole domain of time and at large times the Rayleigh number approaches the well-known value of $Ra_{q,c} = 1296$.

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